Solution 10.1

(a) The time-independent Schrödinger equation for a particle mass m is

$$\hat{H}^{(0)}\psi_n^{(0)}(x, y, z) = E_n^{(0)}\psi_n^{(0)}(x, y, z)$$

where, in this case, the potential $V(x, y, z)$ is infinite except in a region $0 < x < L$, $0 < y < L$, and $0 < z < L$ where $V(x, y, z) = 0$. Hence,

$$\hat{H}^{(0)} = \frac{-\hbar^2}{2m}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)$$

in the region $0 < x < L$, $0 < y < L$, $0 < z < L$. Each eigenstate $\psi_n^{(0)}(x, y, z)$ is separable such that $\psi_n^{(0)}(x, y, z) = \phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$. The functions $\phi_{n_x}(x)$, $\phi_{n_y}(y)$, and $\phi_{n_z}(z)$ are of the form

$$\phi_{n_x}(x) = \sqrt{\frac{2}{L}}\sin(k_{n_x}x)$$

where $k_{n_x} = \frac{n_x\pi}{L}$ and n_x is a non-zero positive integer. Hence, the eigenfunctions for the unperturbed system are

$$\psi^{(0)}_{n_x, n_y, n_z}(x, y, z) = \left(\frac{2}{L}\right)^{3/2}\sin(k_{n_x}x)\sin(k_{n_y}y)\sin(k_{n_z}y)$$

(b) Substituting the eigenfunctions into the time-independent Schrödinger equation gives eigenenergies

$$E^{(0)}_{n_x, n_y, n_z} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$$

where n_x, n_y, and n_z are non-zero positive integers.

The degeneracy of the ground state (1, 1, 1) is one and the degeneracy of the first excited state ((1,1,2), (1,2,1), (2,1,1)) is three.

(c) The system is perturbed by introducing a potential $\hat{W} = V_0$ in a region for which $0 < x < \frac{L}{2}$, $0 < y < \frac{L}{2}$, and $0 < z < L$. The perturbation $\hat{W} = 0$ elsewhere and V_0 is a constant. The new ground state energy is given by the diagonal matrix element of *non-degenerate* perturbation theory. Hence,

$$E^{(1)}_{111} = \langle\psi^{(0)}_{111}|\hat{W}|\psi^{(0)}_{111}\rangle = \left(\frac{2}{L}\right)^3 V_0\int_0^{L/2}\sin^2\left(\frac{\pi x}{L}\right)dx\int_0^{L/2}\sin^2\left(\frac{\pi y}{L}\right)dy\int_0^{L}\sin^2\left(\frac{\pi z}{L}\right)dz$$

$$= \left(\frac{2}{L}\right)^3 V_0\left(\frac{L}{4}\right)\left(\frac{L}{4}\right)\left(\frac{L}{2}\right) = \frac{V_0}{4}$$

(d) The first excited state is three-fold degenerate and so we must use *degenerate* perturbation theory. The 3×3 sub-matrix in the basis $\{\psi^{(0)}_{112}, \psi^{(0)}_{121}, \psi^{(0)}_{211}\}$ is

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} = \begin{bmatrix} \langle\psi_{112}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle & \langle\psi_{112}^{(0)}|\hat{W}|\psi_{121}^{(0)}\rangle & \langle\psi_{112}^{(0)}|\hat{W}|\psi_{211}^{(0)}\rangle \\ \langle\psi_{121}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle & \langle\psi_{121}^{(0)}|\hat{W}|\psi_{121}^{(0)}\rangle & \langle\psi_{121}^{(0)}|\hat{W}|\psi_{211}^{(0)}\rangle \\ \langle\psi_{211}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle & \langle\psi_{211}^{(0)}|\hat{W}|\psi_{121}^{(0)}\rangle & \langle\psi_{211}^{(0)}|\hat{W}|\psi_{211}^{(0)}\rangle \end{bmatrix}$$

The diagonal matrix elements are

$$W_{11} = W_{22} = W_{33} = \langle\psi_{112}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle$$

$$= \left(\frac{2}{L}\right)^3 V_0 \int_0^{L/2} \sin^2\left(\frac{\pi x}{L}\right) dx \int_0^{L/2} \sin^2\left(\frac{\pi y}{L}\right) dy \int_0^{L} \sin^2\left(\frac{2\pi z}{L}\right) dz$$

$$= \left(\frac{2}{L}\right)^3 V_0 \left(\frac{L}{4}\right)\left(\frac{L}{4}\right)\left(\frac{L}{2}\right) = \frac{V_0}{4}$$

The off-diagonal elements

$$W_{12} = W_{21} = \langle\psi_{112}^{(0)}|\hat{W}|\psi_{121}^{(0)}\rangle = \langle\psi_{121}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle$$

$$= \left(\frac{2}{L}\right)^3 V_0 \int_0^{L/2} \sin^2\left(\frac{\pi x}{L}\right) dx \int_0^{L/2} \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{2\pi y}{L}\right) dy \int_0^{L} \sin\left(\frac{\pi z}{L}\right) \sin\left(\frac{2\pi z}{L}\right) dz = 0$$

and

$$W_{13} = W_{31} = \langle\psi_{112}^{(0)}|\hat{W}|\psi_{211}^{(0)}\rangle = \langle\psi_{211}^{(0)}|\hat{W}|\psi_{112}^{(0)}\rangle$$

$$= \left(\frac{2}{L}\right)^3 V_0 \int_0^{L/2} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx \int_0^{L/2} \sin^2\left(\frac{2\pi y}{L}\right) dy \int_0^{L} \sin\left(\frac{\pi z}{L}\right) \sin\left(\frac{2\pi z}{L}\right) dz = 0$$

and

$$W_{23} = W_{32} = \langle\psi_{121}^{(0)}|\hat{W}|\psi_{211}^{(0)}\rangle = \langle\psi_{211}^{(0)}|\hat{W}|\psi_{121}^{(0)}\rangle$$

$$= \left(\frac{2}{L}\right)^3 V_0 \int_0^{L/2} \sin\left(\frac{2\pi x}{L}\right) \sin\left(\frac{\pi x}{L}\right) dx \int_0^{L/2} \sin\left(\frac{\pi y}{L}\right) \sin\left(\frac{2\pi y}{L}\right) dy \int_0^{L} \sin^2\left(\frac{\pi z}{L}\right) dz$$

which may be rewitten as

$$W_{12} = \left(\frac{4V_0}{L^2}\right)\frac{1}{2}\int_0^{L/2} \left(\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right) dx \, \frac{1}{2}\int_0^{L/2} \left(\cos\left(\frac{-\pi y}{L}\right) - \cos\left(\frac{3\pi y}{L}\right)\right) dy$$

where we used the fact that $\int_0^L \sin^2\left(\frac{\pi z}{L}\right) dz = \frac{L}{2}$. Performing the integral

$$W_{12} = \left(\frac{4V_0}{L^2}\right)\frac{1}{4}\left[\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi x}{L}\right)\right]_0^{L/2} \left[\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi x}{L}\right)\right]_0^{L/2}$$

$$= \frac{V_0}{L^2}\left(\frac{L}{\pi} + \frac{L}{3\pi}\right)\left(\frac{L}{\pi} + \frac{L}{3\pi}\right) = \frac{V_0}{L^2}\left(\frac{4L}{3\pi}\right)\left(\frac{4L}{3\pi}\right) = \frac{16V_0}{9\pi^2}$$

Hence,

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} = \frac{V_0}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{64}{9\pi^2} \\ 0 & \frac{64}{9\pi^2} & 1 \end{bmatrix}$$

We seek solutions to

$$(\boldsymbol{H} - E\mathbf{1})\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = (\boldsymbol{H}^{(0)} + \mathbf{W} - E\mathbf{1})\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{V_0}{4}\begin{bmatrix} 1-E & 0 & 0 \\ 0 & 1-E & \frac{64}{9\pi^2} \\ 0 & \frac{64}{9\pi^2} & 1-E \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

which has characteristic equation

$$(1-E)((1-E)(1-E) - \Delta^2) = (1-E)(E^2 - 2E + (1-\Delta)) = 0$$

with roots

$$E = 1$$

and

$$E_{\pm} = \frac{2 \pm \sqrt{4 - 4(1-\Delta^2)}}{2} = 1 \pm \Delta$$

where we set $\Delta = \frac{64}{9\pi^2}$

The new energy levels are

$$E = E_{112}^{(0)} + \frac{V_0}{4}$$

with eigenstate

$$\psi = \psi_{112}^{(0)}$$

and

$$E = E_{112}^{(0)} + \frac{V_0}{4}\left(1 + \frac{64}{9\pi^2}\right)$$

with eigenstate

$$\psi = \frac{1}{\sqrt{2}}(\psi_{121}^{(0)} + \psi_{211}^{(0)})$$

and

$$E = E_{112}^{(0)} + \frac{V_0}{4}\left(1 - \frac{64}{9\pi^2}\right)$$

with eigenstate

$$\psi = \frac{1}{\sqrt{2}}(\psi_{121}^{(0)} - \psi_{211}^{(0)})$$

The orginally degenerate energy levels of the first excited state split into new energy levels because when the perturbation is turned on there are contributions from both diagonal and off-diagonal matrix elements.

Solution 10.2

(a) The exact eigenvalues of $\boldsymbol{H}$ are found by solving the secular equation

$$|\boldsymbol{H}^{(0)} + \boldsymbol{W} - \mathbf{1}E| = \begin{vmatrix} 1-E & \Delta & 0 \\ \Delta & 3-E & 0 \\ 0 & 0 & 2+\Delta-E \end{vmatrix} = 0$$

$$((1-E)(3-E)-\Delta^2)(2+\Delta-E) = (E^2-4E+3-\Delta^2)(2+\Delta-E) = 0$$

has solutions

$$E = \frac{4 \pm \sqrt{16-4(3-\Delta^2)}}{2} = 2 \pm \sqrt{1+\Delta^2}$$

and

$$E = 2+\Delta$$

where we used the fact that $ax^2+bx+c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2+4ac}}{2a}$.

Hence, the exact eigenvalues are

$$E_1 = 2+\sqrt{1+\Delta^2}$$

$$E_2 = 2-\sqrt{1+\Delta^2}$$

$$E_3 = 2+\Delta$$

The eigenfunctions are found by substituting the eigenvalues into the matrix equation. For eigenvalue $E_1 = 2+\sqrt{1+\Delta^2}$ we have

$$\begin{bmatrix} 1-(2+\sqrt{1+\Delta^2}) & \Delta & 0 \\ \Delta & 3-(2+\sqrt{1+\Delta^2}) & 0 \\ 0 & 0 & 2+\Delta-(2+\sqrt{1+\Delta^2}) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

so that

$$(-1-\sqrt{1+\Delta^2})a_1 + \Delta a_2 = 0$$

$$\Delta a_1 + (1-\sqrt{1+\Delta^2})a_2 = 0$$

$$(\Delta - \sqrt{1+\Delta^2})a_3 = 0$$

Let $a_3 = 0$ and $a_1 = 1$ then $a_2 = \frac{-\Delta a_1}{(1-\sqrt{1+\Delta^2})}$. We can check this result by substituting back into the matrix equation to give

$$(-1-\sqrt{1+\Delta^2}) - \frac{\Delta^2}{(1-\sqrt{1+\Delta^2})} + \Delta - \frac{(1-\sqrt{1+\Delta^2})}{(1-\sqrt{1+\Delta^2})}\Delta = 0$$

$$-(1+\sqrt{1+\Delta^2})(1-\sqrt{1+\Delta^2}) - \Delta^2 = 0$$

$$(1+\sqrt{1+\Delta^2})(1-\sqrt{1+\Delta^2}) + \Delta^2 = 0$$

$$1+\sqrt{1+\Delta^2} - \sqrt{1+\Delta^2} - 1 - \Delta^2 + \Delta^2 = 0$$

Similarly for eigenvalue $E_2 = 2-\sqrt{1+\Delta^2}$ we have

$$\begin{bmatrix} 1-(2-\sqrt{1+\Delta^2}) & \Delta & 0 \\ \Delta & 3-(2-\sqrt{1+\Delta^2}) & 0 \\ 0 & 0 & 2+\Delta-(2-\sqrt{1+\Delta^2}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$$

so that

$$(-3+\sqrt{1+\Delta^2})a_1 + \Delta a_2 = 0$$

$$\Delta a_1 + (1+\sqrt{1+\Delta^2})a_2 = 0$$

$$(\Delta+\sqrt{1+\Delta^2})a_3 = 0$$

Let $a_3 = 0$ and $a_1 = 1$ then $a_2 = \dfrac{-\Delta a_1}{(1+\sqrt{1+\Delta^2})}$. We can check this result by substituting back into the matrix equation to give

$$(-1+\sqrt{1+\Delta^2}) - \frac{\Delta^2}{(1+\sqrt{1+\Delta^2})} + \Delta - \frac{(1+\sqrt{1+\Delta^2})}{(1+\sqrt{1+\Delta^2})}\Delta = 0$$

$$(-1+\sqrt{1+\Delta^2})(1+\sqrt{1+\Delta^2}) - \Delta^2 = 0$$

$$-1+\sqrt{1+\Delta^2}^2 - \sqrt{1+\Delta^2}^2 + 1 - \Delta^2 + \Delta^2 = 0$$

So the three eigenstates ϕ_1, ϕ_2, ϕ_3 associated with eigenvalues E_1, E_2, and E_3 respectively, are

$$\phi_1 = \begin{bmatrix} 1 \\ \dfrac{-\Delta}{(1-\sqrt{1+\Delta^2})} \\ 0 \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} 1 \\ \dfrac{-\Delta}{(1+\sqrt{1+\Delta^2})} \\ 0 \end{bmatrix}$$

$$\phi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The normalization constant A for the state ϕ_1 proceeds by finding the value of A such that $A^2\phi_1^T\phi_1 = 1$. To check the orthogonality of ϕ_1 and ϕ_2 we need to show that $\phi_1^T\phi_2 = 0$.

$$\begin{bmatrix} 1 & \dfrac{-\Delta}{(1-\sqrt{1+\Delta^2})} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \dfrac{-\Delta}{(1+\sqrt{1+\Delta^2})} \\ 0 \end{bmatrix} = 1 + \frac{\Delta^2}{(1-\sqrt{1+\Delta^2})(1+\sqrt{1+\Delta^2})}$$

$$1 + \frac{\Delta^2}{(1-(1+\Delta^2))} = 1 - \frac{\Delta^2}{\Delta^2} = 0$$

(b) The eigenvalues of the perturbed system to second-order in time-independent perturbation theory are

$$E = E_m^{(0)} + W_{mm} + \sum_{n \neq m} \frac{|W_{mn}|^2}{E_m^{(0)} - E_n^{(0)}}$$

The values of $E_m^{(0)}$ are just the eigenvalues for the unperturbed states. These are obtained directly from the diagonal marix elements of the unperturbed Hamiltonian

$$\hat{H}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and so $E_1^{(0)} = 1$, $E_2^{(0)} = 3$, and $E_3^{(0)} = 2$.

The first-order correction to the energy eigenvalues is given by the diagonal matrix elements of the perturbation

$$\hat{W} = \begin{bmatrix} 0 & \Delta & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & \Delta \end{bmatrix}$$

and so $E_1^{(1)} = 0$, $E_2^{(1)} = 0$, and $E_3^{(1)} = \Delta$.

The second-order correction to the energy eigenvalues is given by

$$E_1^{(2)} = \frac{|W_{12}|^2}{E_1^{(0)} - E_2^{(0)}} + \frac{|W_{13}|^2}{E_1^{(0)} - E_3^{(0)}} = \frac{\Delta^2}{1-3} + \frac{0}{1-2} = \frac{-\Delta^2}{2}$$

$$E_2^{(2)} = \frac{|W_{21}|^2}{E_2^{(0)} - E_1^{(0)}} + \frac{|W_{23}|^2}{E_2^{(0)} - E_3^{(0)}} = \frac{\Delta^2}{3-1} + \frac{0}{3-2} = \frac{\Delta^2}{2}$$

$$E_3^{(2)} = \frac{|W_{31}|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{|W_{32}|^2}{E_3^{(0)} - E_2^{(0)}} = \frac{0}{2-1} + \frac{0}{2-3} = 0$$

We can now write down the eigenvalues of the perturbed system to second-order in time-independent perturbation theory. They are

$$E_1 = 1 - \frac{\Delta^2}{2}$$

$$E_2 = 3 + \frac{\Delta^2}{2}$$

$$E_3 = 2 + \Delta$$

(c) If we expand the expression for the exact result in a binomial series we obtain

$$E = 2 \pm \sqrt{1 + \Delta^2} = 2 \pm \left(1 + \frac{\Delta^2}{2} + \ldots\right)$$

giving $E_1 = 1 - \frac{\Delta^2}{2}$ and $E_2 = 3 + \frac{\Delta^2}{2}$. So we may conclude that second order perturbation theory (b) are in agreement with the exact results of (a).

Solution 10.4

(a) For the box potential the eigenenergies are $E_n = \frac{\pi^2\hbar^2 n^2}{2m^* L^2}$ and because the energy of the electron is given as $E_n = \frac{6\pi^2\hbar^2}{2m^* L^2}$ we have $n^2 = 6$. The number n^2 satisfies $n^2 = n_x^2 + n_y^2 + n_z^2 = 6$ which can only be true for ($n_x = 1$, $n_y = 1$, $n_z = 2$), ($n_x = 1$, $n_y = 2$, $n_z = 1$), and ($n_x = 2$, $n_y = 1$, $n_z = 1$). Hence the degeneracy is 3 and the three wave functions are

$$\psi_{1,1,2}(x, y, z) = \sqrt{\frac{8}{L^3}}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{2\pi z}{L}\right)$$

$$\psi_{1,2,1}(x, y, z) = \sqrt{\frac{8}{L^3}}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

$$\psi_{2,1,1}(x, y, z) = \sqrt{\frac{8}{L^3}}\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

The time-independent Schrödinger equation for the unperturbed system in matrix form is

$$(\boldsymbol{H} - E\mathbf{1})\boldsymbol{a} = \begin{bmatrix} H_{11} - E & 0 & 0 \\ 0 & H_{22} - E & 0 \\ 0 & 0 & H_{33} - E \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The perturbing potential is

$$W = e|\mathbf{E}|z$$

In matrix form we have

$$\mathbf{W} = e|\mathbf{E}|\begin{bmatrix} \langle 1,1,2|z|1,1,2\rangle & \langle 1,1,2|z|1,2,1\rangle & \langle 1,1,2|z|2,1,1\rangle \\ \langle 1,2,1|z|1,1,2\rangle & \langle 1,2,1|z|1,2,1\rangle & \langle 1,2,1|z|2,1,1\rangle \\ \langle 2,1,1|z|1,1,2\rangle & \langle 2,1,1|z|1,2,1\rangle & \langle 2,1,1|z|2,1,1\rangle \end{bmatrix}$$

The diagonal matrix elements are found noting

$$\langle 1,1,2|z|1,1,2\rangle = \langle 1,2,1|z|1,2,1\rangle = \langle 2,1,1|z|2,1,1\rangle$$

and solving

$$\langle 2,1,1|z|2,1,1\rangle = \frac{8}{L^3}\int_0^L \sin^2\left(\frac{2\pi x}{L}\right)dx\int_0^L \sin^2\left(\frac{\pi y}{L}\right)dy\int_0^L z\sin^2\left(\frac{\pi z}{L}\right)dz = \frac{2}{L}\int_0^L z\sin^2\left(\frac{\pi z}{L}\right)dz = \frac{L}{2}$$

The off-diagonal matrix elements

$$\langle 1,2,1|z|1,1,2\rangle = \langle 2,1,1|z|1,2,1\rangle = \langle 2,1,1|z|1,1,2\rangle = 0$$

and so the new energy eigenvalues are

$$E = \frac{3\pi^2\hbar^2}{m^* L^2} + \frac{e|\mathbf{E}|L}{2}$$

(b) Using $m^* = 0.07 \times m_0$ we obtain an unperturbed energy level value of

$$E = \frac{3\pi^2\hbar^2}{0.07\times m_0\times L^2} = \frac{3\pi^2(1.05\times10^{-34})^2}{0.07\times9.1\times10^{-31}\times(2\times10^{-8})^2} = 80\ \text{meV}$$

The perturbation due to application of an electric field of strength $|\mathbf{E}| = 10^4\ \text{V cm}^{-1} = 10^6\ \text{V m}^{-1}$ shifts this level by

$$\frac{e|\mathbf{E}|L}{2} = \frac{1.6\times10^{-19}\times10^6\times20\times10^{-9}}{2} = 10\ \text{meV}$$

Solution 10.5

(a) The eigenenergies of the unperturbed Hamiltonian are $E_n^{(0)} = \hbar\omega_0\left(n+\frac{1}{2}\right)$ for $n = 0, 1, 2, \ldots$. The first-order correction is $E^{(1)} = W_{nn}$ where $W_{nn} = \langle n|\hat{W}|n\rangle$ so that the new energy eigenvalues to first-order are $E_n = E_n^{(0)} + W_{nn}$.

$$E_0 = \frac{\hbar\omega_0}{2} + \Delta\hbar\omega_0 = \frac{\hbar\omega_0}{2}(1+2\Delta)$$

$$E_1 = \frac{3\hbar\omega_0}{2}$$

$$E_2 = \frac{5\hbar\omega_0}{2} + \frac{\Delta\hbar\omega_0}{2} = \frac{\hbar\omega_0}{2}(5+\Delta)$$

$$E_3 = \frac{7\hbar\omega_0}{2}$$

(b) The new energy eigenvalues to second-order are given by

$$E_n = E_n^{(0)} + W_{nn} + \sum_{m\neq n}\frac{|W_{nm}|^2}{E_n^{(0)} - E_m^{(0)}}$$

and so

$$E_0 = \frac{\hbar\omega_0}{2} + \Delta\hbar\omega_0 - \frac{\Delta^2\hbar\omega_0}{4}$$

$$E_1 = \frac{3\hbar\omega_0}{2}$$

$$E_2 = \frac{5\hbar\omega_0}{2} + \frac{\Delta\hbar\omega_0}{2} + \frac{\Delta^2\hbar\omega_0}{4}$$

$$E_3 = \frac{7\hbar\omega_0}{2}$$